

New Expressions for Ergodic Capacities of Optical Fibers and Wireless MIMO Channels

Amor Nafkha, Nizar Demni, Rémi Bonnefoi

Abstract

Multimode/multicore fibers are expected to provide an attractive solution to overcome the capacity limit of current optical communication system. In presence of high crosstalk between modes/cores, the squared singular values of the input/output transfer matrix follow the law of the Jacobi ensemble of random matrices. Assuming that the channel state information is only available at the receiver, we derive in this paper a new expression for the ergodic capacity of the Jacobi MIMO channel. This expression involves double integrals which can be evaluated easily and efficiently. Moreover, the method used in deriving this expression does not appeal to the classical one-point correlation function of the random matrix model. Using a limiting transition between Jacobi and Laguerre polynomials, we derive a similar formula for the ergodic capacity of the Gaussian MIMO channel. The analytical results are compared with Monte Carlo simulations and related results available in the literature. A perfect agreement is obtained.

Index Terms

Jacobi MIMO channel, Gaussian MIMO channel, Jacobi polynomials, Laguerre polynomials, Ergodic capacity.

I. INTRODUCTION

To accommodate the exponential growth of data traffic over the last few years, the space-division multiplexing (SDM) based on multi-core optical fiber (MCF) or multi-mode optical fiber (MMF) is expected to overcome the barrier from capacity limit of single-core fiber [1]–[3]. The main challenge in SDM occurs due to in-band crosstalk between multiple parallel transmission channels (cores/modes). This non-negligible crosstalk can be dealt with using multiple-input multiple-output (MIMO) signal processing techniques [2], [4]–[7]. Those techniques are widely used for wireless communication systems and they

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helped to drastically increase channel capacity. Assuming important crosstalk between cores and/or modes, negligible backscattering and near-lossless propagation, we can model the transmission optical channel as a random complex unitary matrix [8]–[10].

In [8], authors appealed to the Jacobi unitary ensemble (JUE) to establish the propagation channel model for MIMO communications over multi-mode/multi-core optical fibers. The JUE is a matrix-variate analogue of the beta random variable and consists of complex Hermitian random matrices which can be realized at least in two different ways [12], [13]. One of them mimics the construction of the beta random variable as a ratio of two independent Gamma random variables: the latter are replaced by two independent complex Hermitian Wishart matrices whose sum is invertible. Otherwise, one draws a Haar-distributed unitary matrix then takes the square of the radial part of an upper left corner [12]. By a known fact for unitarily invariant-random matrices [13], the average of any symmetric function with respect to the eigenvalues density can be expressed through the one-point correlation function, also known as the single-particle density. In particular, the ergodic capacity of a matrix drawn from the JUE can be represented by an integral where the integrand involves the Christoffel-Darboux kernel associated with Jacobi polynomials ([13], p.384). The drawback of this representation is the dependence of this kernel on the size of the matrix. Indeed, its diagonal is written either as a sum of squares of Jacobi polynomials and the number of terms in this sum equals the size of the matrix least one, or by means of the Christoffel-Darboux formula as a difference of the product of two Jacobi polynomials whose degrees depend on the size of the matrix. To the best of our knowledge, this is the first study that derives exact expression of the ergodic capacity as a double integral over a suitable region.

In this paper, we provide a new expression for the ergodic capacity of the Jacobi MIMO channel relying this time on the formula derived in [15] for the moments of the eigenvalues density of the Jacobi random matrix. The obtained expression shows that the ergodic capacity is an average of some function over the signal-to-noise ratio, and it has the merit to have a simple dependence on the size of the matrix which allows for easier and more precise numerical simulations. By a limiting transition between Jacobi and Laguerre polynomials [16], we derive a similar expression for the ergodic capacity of the Gaussian MIMO channel [17].

The paper is organized as follows. In Section II, we settle down some notations and recall the definitions of random matrices and special functions occurring in the remainder of the paper. Section III presents the system model. The main results of this paper are presented in section IV and are illustrated in section VI by numerical simulations followed by several comments. Finally, the proofs of these results are provided in appendices.

II. DEFINITIONS AND NOTATIONS

Throughout this paper, the following notations and definitions are used. We start with those concerned with special functions for which the reader is referred to the standard book [16]. The Pochhammer symbol $(x)_k$ with $x \in \mathbb{R}$ and $k \in \mathbb{N}$ is defined by

$$(x)_k = x(x+1) \dots (x+k-1); (x)_0 = 1 \quad (1)$$

For $x > 0$, it is clear that

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} \quad (2)$$

where $\Gamma(\cdot)$ is the Gamma function. Note that if $x = -q$ is a non positive integer then

$$(-q)_k = \begin{cases} (-1)^k \frac{q!}{(q-k)!} & \text{if } k \geq q \\ 0 & \text{if } k < q \end{cases} \quad (3)$$

Next, the Gauss hypergeometric function ${}_2F_1$ is defined for complex $|z| < 1$ by the convergent power series

$${}_2F_1(\theta, \sigma, \gamma, z) = \sum_{k=0}^{\infty} \frac{(\theta)_k (\sigma)_k}{(\gamma)_k k!} z^k \quad (4)$$

where $(\cdot)_k$ denotes the Pochhammer symbol defined in (1) and θ, σ, γ are real parameters with $\gamma \neq 0, -1, -2, \dots$. The function ${}_2F_1$ has an analytic continuation to the complex plane cut along the half-line $[1, \infty[$. In particular, the Jacobi polynomials $P_q^{\alpha, \beta}(x)$ of degree q and parameters $\alpha > -1, \beta > -1$ can also be expressed in terms of the Gauss hypergeometric function (4) as follows

$$P_q^{\alpha, \beta}(x) = \frac{(\alpha+1)_q}{q!} {}_2F_1(-q, q+\alpha+\beta+1, \alpha+1; (1-x)/2) \quad (5)$$

An important asymptotic property of the Jacobi polynomial is that it can be reduced to the q -th Laguerre polynomial of parameter α through the following limit

$$L_q^\alpha(x) = \lim_{\beta \rightarrow \infty} P_q^{\alpha, \beta} \left(1 - \frac{2x}{\beta} \right), \quad x > 0 \quad (6)$$

Now, we come to the notations and the definitions related with random matrices, and refer the reader to [12]–[14]. Firstly, the Hermitian transpose and the determinant of a complex matrix \mathbf{A} are denoted by \mathbf{A}^\dagger and $\det(\mathbf{A})$ respectively. Secondly, the Laguerre unitary ensemble (LUE) is formed out of non negative definite matrices $\mathbf{A}^\dagger \mathbf{A}$ where \mathbf{A} is a rectangular $m \times n$ matrix, with $m \geq n$, whose entries are complex independent Gaussian random variables. A matrix from the LUE is often referred to as a complex Wishart matrix and (m, n) are its degrees of freedom and its size respectively. Finally, let

$\mathbf{X} = \mathbf{A}^\dagger \mathbf{A}$ and $\mathbf{Y} = \mathbf{B}^\dagger \mathbf{B}$ be two independent (m_1, n) and (m_2, n) complex Wishart matrices. Assume $m_1 + m_2 \geq n$, then $\mathbf{X} + \mathbf{Y}$ is positive definite and

$$\mathbf{J} = (\mathbf{X} + \mathbf{Y})^{-1/2} \mathbf{X} (\mathbf{X} + \mathbf{Y})^{-1/2}$$

belongs to the JUE. The matrix \mathbf{J} is unitarily-invariant and satisfies $\mathbf{0}_n \leq \mathbf{J} \leq \mathbf{I}_n$ where $\mathbf{0}_n, \mathbf{I}_n$ stand for the null and the identity matrices respectively¹. If $m_1, m_2 \geq n$ then \mathbf{J} and $\mathbf{I}_n - \mathbf{J}$ are positive definite and the joint distribution of the ordered eigenvalues of \mathbf{J} has a density given by

$$\mathcal{F}_{a,b,n}(\lambda_1, \dots, \lambda_n) = Z_{a,b,n}^{-1} \prod_{1 \leq j \leq n} \lambda_j^{a-1} (1 - \lambda_j)^{b-1} [V(\lambda_1, \dots, \lambda_n)]^2 \mathbf{1}_{0 < \lambda_1 < \dots < \lambda_n < 1} \quad (7)$$

with respect to Lebesgue measure $d\lambda = d\lambda_1 \dots d\lambda_n$. Here, $a = m_1 - n + 1, b = m_2 - n + 1$, $Z_{a,b,n}$ is a normalization constant read off from the Selberg integral [14], [15]:

$$Z_{a,b,n} = \prod_{j=1}^n \frac{\Gamma(a + j - 1) \Gamma(b + j - 1) \Gamma(1 + j)}{\Gamma(a + b + n + j - 2)}$$

and

$$V(\lambda_1, \dots, \lambda_n) = \prod_{1 \leq j < k \leq n} (\lambda_j - \lambda_k)$$

is the Vandermonde polynomial.

Another construction of matrices from the JUE is as follows [12]: let \mathbf{U} be an $m \times m$ Haar-distributed unitary matrix. Let m_r and m_t be two positive integers such that $m_r + m_t \leq m$ and $m_t \leq m_r$. Let also \mathbf{H} be the $m_r \times m_t$ upper-left corner of \mathbf{U} , then the joint distribution of the ordered eigenvalues of $\mathbf{H}^\dagger \mathbf{H}$ is given by (7) with parameters $a = m_r - m_t + 1$, $b = m - m_r - m_t + 1$, and $n = m_t$.

III. SYSTEM MODEL

We consider an optical space-division multiplexing where the multiple channels correspond to the number of excited modes/cores within the optical fiber. The coupling between different modes and/or cores can be described by scattering matrix formalism [10], [18], [19]. In this paper, we consider m -channel lossless optical fiber with $m_t \leq m$ transmitting excited channels and $m_r \leq m$ receiving channels. The scattering matrix formalism can describe very simply the propagation through the fiber using $2m \times 2m$ scattering matrix \mathbf{S} given as

$$\mathbf{S} = \begin{bmatrix} \mathbf{r}_l & \mathbf{t}_r \\ \mathbf{t}_l & \mathbf{r}_r \end{bmatrix}, \quad (8)$$

¹For two square matrices A and B , we write $A \leq B$ when $B - A$ is a non negative matrix.

where the $m \times m$ block matrices \mathbf{r}_l and \mathbf{r}_r describe the reflection from left to left and from right to right of the fiber, respectively, and \mathbf{t}_l and \mathbf{t}_r describe the transmission through the fiber from left to right and from right to left, respectively. Since the fiber is assumed to be lossless and time-reversal, the scattering matrix must be a complex unitary symmetric matrix, (*i.e.* $\mathbf{S}^\dagger \mathbf{S} = \mathbf{I}_{2m}$). Therefore, the four Hermitian matrices $\mathbf{t}_l \mathbf{t}_l^\dagger$, $\mathbf{t}_r \mathbf{t}_r^\dagger$, $\mathbf{I}_m - \mathbf{r}_r \mathbf{r}_r^\dagger$, and $\mathbf{I}_m - \mathbf{r}_l \mathbf{r}_l^\dagger$ have the same set of eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$. Each of these m transmission eigenvalues is a real number belong to the interval $[0, 1]$. Assuming a unitary coupling among all transmission modes the overall transfer matrix \mathbf{t}_l can be described by a $m \times m$ unitary matrix, where each entry t_{ij} represents the complex path gain from transmitted mode i to received mode j . Moreover, the transmission matrix \mathbf{t}_l is Haar distributed over the group of complex unitary matrices [8], [10]. Given the fact that only $m_t \leq m$ and $m_r \leq m$ modes are addressed by the transmitter and receiver, respectively, the effective transmission channel matrix $\mathbf{H} \in \mathbb{C}^{m_r \times m_t}$ is a truncated² version of \mathbf{t}_l . As a result, the corresponding MIMO channel for this system reads

$$\mathbf{y} = \sqrt{\rho} \mathbf{H} \mathbf{x} + \mathbf{z} \quad (9)$$

where $\mathbf{x} \in \mathbb{C}^{m_t \times 1}$ is the transmitted vector, $\mathbf{y} \in \mathbb{C}^{m_r \times 1}$ is the received vector, and $\mathbf{z} \in \mathbb{C}^{m_r \times 1}$ is the additive white circularly symmetric complex Gaussian noise vector normalized with covariance matrix equal to the identity matrix \mathbf{I}_{m_r} .

IV. THE ERGODIC CAPACITY OF THE JACOBI MIMO CHANNEL

In this section, we provide a new expression for the ergodic capacity in the setting of Jacobi fading channels. We assume that the channel state information (CSI) is known only at the receiver, not at the transmitter. The channel ergodic capacity, under a total average transmit power constraint, is then achieved by taking \mathbf{x} as a vector of zero-mean circularly symmetric complex Gaussian components with covariance matrix $\rho \mathbf{I}_{m_t}/m_t$, and it is given by [8, Eq. (10)]

$$\begin{aligned} C(m_t, m_r, m, \rho) &= \mathbb{E}_{\mathbf{H}} \left[\ln \det \left(\mathbf{I}_{m_t} + \frac{\rho}{m_t} \mathbf{H}^\dagger \mathbf{H} \right) \right] \\ &= \mathbb{E}_{\mathbf{H}} \left[\ln \det \left(\mathbf{I}_{m_r} + \frac{\rho}{m_t} \mathbf{H} \mathbf{H}^\dagger \right) \right] \end{aligned} \quad (10)$$

where $\mathbb{E}_{\mathbf{H}}$ denotes the expectation over all channel realizations. Without loss of generality, we shall assume in the sequel that $m_t \leq m_r$ and $m \geq m_r + m_t$. Indeed, $\mathbf{H}^\dagger \mathbf{H}$ and $\mathbf{H} \mathbf{H}^\dagger$ share the same non zero

²Without loss of generality, the effective transmission channel matrix \mathbf{H} is the $m_r \times m_t$ upper-left corner of the transmission matrix \mathbf{t}_l [12], [20]

eigenvalues while if $m < m_r + m_t$, then $(m - m_r) + (m - m_t) < m$ and [8, Theorem 2] shows that the ergodic capacity is given by

$$C(m_t, m_r, m, \rho) = (m_t + m_r - m)C(1, 1, 1, \rho) + C(m - m_r, m - m_t, m, \rho). \quad (11)$$

For our purposes, we assume further that $m > m_t + m_r \Leftrightarrow b \geq 2$ and the case $m = m_r + m_t \Leftrightarrow b = 1$ can be dealt with by a limiting procedure. Actually, our formula for the ergodic capacity derived below is valid for real $a > 0, b > 1$ and we can consider its limit as $b \rightarrow 1$. However, for ease of reading, we postpone the details of the computations relative to this limiting procedure to a future forthcoming paper.

Now, recall that the random matrix $\mathbf{H}^\dagger \mathbf{H}$ has the Jacobi distribution and that its ordered eigenvalues have the joint density given by (7) with parameters $a = m_r - m_t + 1$, $b = m - m_t - m_r + 1$. Using (7), the channel ergodic capacity (11) is written explicitly as

$$C(m_t, m_r, m, \rho) = \int_0^1 \dots \int_0^1 \sum_{k=1}^{m_t} \ln(1 + \rho \lambda_k) \mathcal{F}_{a,b,m_t}(\lambda_1, \dots, \lambda_{m_t}) d\lambda_1 \dots d\lambda_{m_t}. \quad (12)$$

A major step towards our main result is the following proposition.

Proposition 1: For any $\rho \in (0, 1)$,

$$[D_\rho(\rho D_\rho)] C(m_t, m_r, m, \rho) = A_{a,b,m_t} \rho^{m_t-1} P_{m_t-1}^{a-1,b} \left(\frac{\rho+2}{\rho} \right) {}_2F_1(m_t+1, a+m_t, a+b+2m_t-1; -\rho)$$

where D_ρ is the derivative operator with respect to ρ and

$$A_{a,b,m_t} = \frac{(a+m_t-1)m_t!}{(a+b+m_t-1)_{m_t}}.$$

Proof: See Appendix A. ■

With this proposition in hand, we are able to derive the following new expression of the ergodic capacity:

Theorem 1: For any $\rho \geq 0$, The ergodic capacity of a Jacobi MIMO fading channel is given by

$$C(m_t, m_r, m, \rho) = -B_{a,b,m_t} \int_0^1 u^{a-1} (1-u)^{b-2} P_{m_t-1}^{a-1,b} (1-2u) P_{m_t}^{a-1,b-2} (1-2u) \left\{ \int_0^\rho \frac{\ln(vu+1)}{v} dv \right\} du, \quad (13)$$

where

$$B_{a,b,m_t} = \frac{m_t! \Gamma(a+b+m_t-1)}{\Gamma(a+m_t-1) \Gamma(m_t+b-1)}.$$

Proof: See Appendix B. ■

V. ERGODIC CAPACITY OF THE GAUSSIAN MIMO CHANNEL

Using the limiting transition (6) between Jacobi and Laguerre polynomials, we give another expression for the ergodic capacity expressions of the Gaussian (wireless) MIMO channels. Indeed, it was shown in [10], [11], that the parameter b in (13) can be interpreted as the power loss through the optical fiber. Therefore, as b becomes large, the channel matrix \mathbf{H} in (9) starts to look like a complex Gaussian matrix with independent and identically distributed entries. As a matter of fact, the Jacobi MIMO channel approaches the Gaussian MIMO channel in the large b -limit corresponding to infinite power loss through the optical fiber. In particular, the ergodic capacity (13) converges as $b \rightarrow \infty$ to the ergodic capacity of the Gaussian MIMO channel already considered by Telatar in [17, Theorem 2] and we obtain:

Theorem 2: The ergodic capacity of the Gaussian MIMO channel with m_t transmitters and m_r receivers is given by

$$C(m_t, m_r, \rho) = -\frac{m_t!}{(m_r - 1)!} \int_0^{+\infty} u^{m_r - m_t} \exp(-u) L_{m_t - 1}^{m_r - m_t}(u) L_{m_t}^{m_r - m_t}(u) \left\{ \int_0^\rho \frac{\ln(vu + 1)}{v} dv \right\} du. \quad (14)$$

Proof: See Appendix C. ■

VI. NUMERICAL RESULTS

In this section, we present numerical results supporting the analytical expressions derived in Section IV and Section V. All Monte Carlo simulation results are obtained with 10^5 runs. Herein, we consider the case where CSI is available at the receiver side. Fig. 1 examines the ergodic capacity of the Jacobi MIMO as a function of the signal-to-noise ratio, when the number of parallel transmission paths is fixed to $m = 20$ and the number of transmit modes equal to the number of receive modes $m_r = m_t$. It is evident that when we increase the number of transmitted and received modes, we improve the ergodic capacity of the system. As expected, the ergodic capacity increases with SNR. Fig. 1 is also shown that the two theoretical expressions curves of the ergodic capacity (13) and [8, (11)] perfectly matched the simulation results.

Fig. 2 shows the theoretical and simulated ergodic capacity of Jacobi MIMO channel as a function of the number of received modes. Here, we fixed the number of parallel transmission paths to $m = 25$, the SNR to $\rho = 10\text{dB}$, and the number of transmit modes m_t to have following values $\{2, 3\}$. It is shown that every simulated curve is in excellent agreement with the theoretical curves calculated from (13) and [8, (11)]. The results show that the capacity increases in a logarithm way with the number of received modes. These results are in line with those obtained for a Gaussian MIMO channel (see Fig. 4).

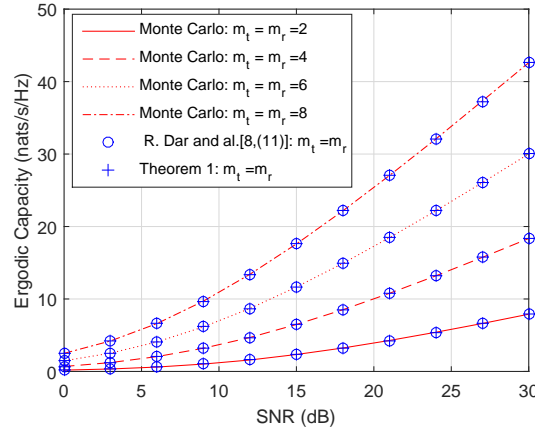


Fig. 1. The variation of the ergodic capacity of the Jacobi MIMO channel as a function of ρ for $m = 20$

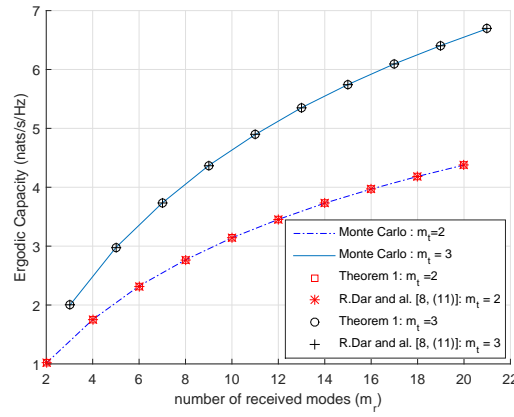


Fig. 2. Ergodic capacity of Jacobi MIMO channel for various numbers of transmit of receive modes, $\rho = 10\text{dB}$, and $m = 25$.

For Gaussian MIMO channel, the proposed expression of the ergodic capacity was verified through Monte Carlo experiments and it is shown in Fig. 3. We can observe that the expression in (14) matches perfectly with the expression introduced by Telatar [17, Eq. (8)]. In Fig. 3, the comparisons are shown between theoretical expressions and simulation values of the ergodic capacity as a function of the SNR. As we can observe in Fig. 3, for a given SNR, the capacity increases as the numbers of transmit and receive antennas grow.

Finally, we show the effects of the number of receive antenna elements on the ergodic capacity of Gaussian MIMO channel in Fig. 4. As expected, we observe the ergodic capacity increases in logarithmic way with increasing numbers of receive antennas. As for optical MIMO channel, the three different ways to compute the Gaussian MIMO capacity give the same results. These simulations were carried out to

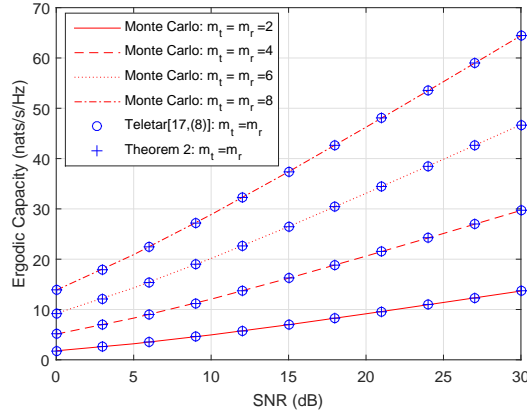


Fig. 3. Ergodic capacity of Gaussian MIMO channel versus SNR for different numbers of transmit and receive antennas.

verify the mathematical development made and no inconsistencies were noted.

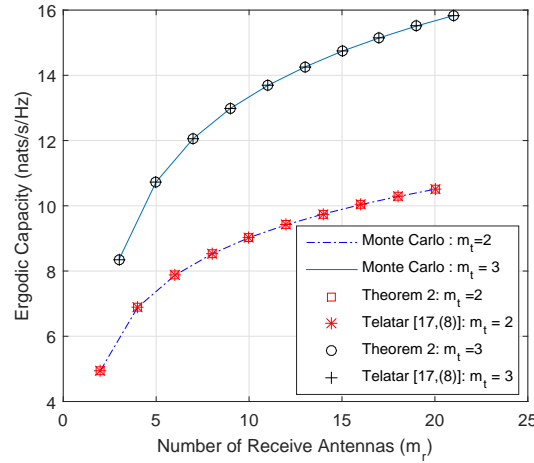


Fig. 4. Evolution of the capacity versus the number of receiving antennas for $\rho = 10dB$

VII. CONCLUSIONS

This paper focused on the Jacobi fading channel and a new expression is proposed for the ergodic capacity. This new expression allows to better understand the role of each of the parameters and, with this formula, numerical evaluation of the capacity does not require the computation of a sum of Jacobi polynomials. This expression was used to make the connection between optical and wireless MIMO channel and allowed to propose a new expression for the capacity of Gaussian MIMO channel. Finally,

numerical simulations were used to verify mathematical derivations and shows the evolution of the capacity versus SNR and versus the number of transmit antennas.

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APPENDIX A

PROOF OF PROPOSITION 1

For ease of reading, we simply denote below the ergodic capacity by $C(\rho)$ and write n for the number of transmitters m_t . Moreover, the reader can easily check that our computations are valid for real $a > 0, b > 1$.

We start by recalling from [15, Corollary 2.3] that for any $k \geq 1$,

$$\int_{\lambda \in [0,1]^n} \left(\sum_{i=1}^n \lambda_i^k \right) \mathcal{F}_{a,b,n}(\lambda) d\lambda = \frac{1}{k!} \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \prod_{j=-i}^{k-i-1} \frac{(n+j)(a+n+j-1)}{(a+b+2n+j-2)}.$$

Now, let $\rho \in [0, 1]$ and use the Taylor expansion

$$\ln(1 + \rho \lambda_i) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(\rho \lambda_i)^k}{k}$$

to get

$$\sum_{i=1}^n \ln(1 + \rho \lambda_i) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\rho^k}{k} \left(\sum_{i=1}^n \lambda_i^k \right).$$

Consequently,

$$C(\rho) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{\rho^k}{k!} \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \prod_{j=-i}^{k-i-1} \frac{(n+j)(a+n+j-1)}{(a+b+2n+j-2)}. \quad (15)$$

Changing the summation order and performing the index change $k \mapsto k + i + 1$ in (15), we get

$$C(\rho) = \sum_{i=0}^{\infty} (-1)^i \sum_{k=0}^{\infty} \frac{(-1)^{k+i}}{(k+i+1)} \frac{\rho^{k+i+1}}{(k+i+1)!} \binom{k+i}{i} \prod_{j=-i}^k \frac{(n+j)(a+n+j-1)}{(a+b+2n+j-2)}.$$

Now, observe that the product displayed in the right hand side of the last equality vanishes whenever $i \geq n$ due to the presence of the factor $j + n, -i \leq j \leq k$. Thus, the first series terminates at $i = n - 1$ and together with the index change $j \mapsto n + j$ in the product lead to

$$C(\rho) = \sum_{i=0}^{n-1} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+i+1)} \frac{\rho^{k+i+1}}{(k+i+1)!} \binom{k+i}{i} \prod_{j=n-i}^{k+n} \frac{(j)(a+j-1)}{(a+b+n+j-2)}.$$

Next, we compute for each $n - i \leq j \leq n + k$

$$\prod_{j=n-i}^{n+k} (j) = \frac{(n+k)!}{(n-i-1)!} = \frac{(n+1)_k n!}{(n-i-1)!},$$

and similarly

$$\begin{aligned} \prod_{j=n-i}^{n+k} (a+j-1) &= \frac{(a)_{n+k}}{(a)_{n-i-1}} = \frac{(a+n)_k (a)_n}{(a)_{n-i-1}} \\ \prod_{j=n-i}^{n+k} (a+b+n+j-2) &= \frac{(a+b+n-1)_{n+k}}{(a+b+n-1)_{n-i-1}} = \frac{(a+b+2n-1)_k (a+b+n-1)_n}{(a+b+n-1)_{n-i-1}}. \end{aligned}$$

Altogether, the ergodic capacity reads

$$\frac{(a)_n}{(a+b+n-1)_n} \sum_{i=0}^{n-1} \frac{n!}{(n-1-i)!i!} \frac{(a+b+n-1)_{n-i-1}}{(a)_{n-i-1}} \sum_{k \geq 0} \frac{(-1)^k \rho^{k+i+1}}{(k+i+1)^2} \frac{(n+1)_k (a+n)_k}{(a+b+2n-1)_k k!}.$$

But the series

$$\sum_{k \geq 0} \frac{(-1)^k \rho^{k+i+1}}{(k+i+1)^2} \frac{(n+1)_k (a+n)_k}{(a+b+2n-1)_k k!}$$

as well as its derivatives with respect to ρ converge uniformly in any closed sub-interval in $]0, 1[$. It follows that

$$\begin{aligned} D_\rho(\rho D_\rho) \sum_{k \geq 0} \frac{(-1)^k \rho^{k+i+1}}{(k+i+1)^2} \frac{(n+1)_k (a+n)_k}{(a+b+2n-1)_k k!} &= \sum_{k \geq 0} (-1)^k \rho^{k+i} \frac{(n+1)_k (a+n)_k}{(a+b+2n-1)_k k!} \\ &= \rho^i {}_2F_1(n+1, a+n, a+b+2n-1; -\rho) \end{aligned}$$

where D_ρ is the derivative operator acting on the variable ρ . Finally, the index change $i \mapsto n - i - 1$ together with

$$(1-n)_i = (1-n+i-1)(1-n+i-2) \dots (1-n) = (-1)^i \frac{(n-1)!}{(n-1-i)!}$$

yield

$$\begin{aligned} \sum_{i=0}^{n-1} \frac{n!}{(n-1-i)!i!} \frac{(a+b+n-1)_{n-i-1}}{(a)_{n-i-1}} \rho^i &= n \sum_{i=0}^{n-1} \frac{(n-1)!}{(n-1-i)!i!} \frac{(a+b+n-1)_i}{(a)_i} \rho^{n-1-i} \\ &= n \rho^{n-1} \sum_{i=0}^{n-1} \frac{(1-n)_i}{i!} \frac{(a+b+n-1)_i}{(a)_i} \left(-\frac{1}{\rho}\right)^i \\ &= n \rho^{n-1} {}_2F_1\left(1-n, a+b+n-1, a; -\frac{1}{\rho}\right) \\ &= \frac{n! \rho^{n-1}}{(a)_{n-1}} P_{n-1}^{a-1, b} \left(\frac{\rho+2}{\rho}\right) \\ &= \frac{n! \rho^{n-1}}{(a)_{n-1}} P_{n-1}^{a-1, b} \left(\frac{\rho+2}{\rho}\right). \end{aligned}$$

Since

$$\frac{n!\rho^{n-1}}{(a)_{n-1}} \frac{(a)_n}{(a+b+n-1)_n} = \frac{n!(a+n-1)\rho^{n-1}}{(a+b+n-1)_n},$$

The statement of the proposition corresponds to the special parameters $a = m_t - m_r + 1$ and $b = m - m_t - m_r + 1$.

APPENDIX B

PROOF OF THEOREM 1

Let $\rho \in [0, 1]$. From [16, Eq. (4.4.6)], we readily deduce that the hypergeometric function

$${}_2F_1(n+1, a+n, a+b+2n-1; -\rho)$$

coincides up to a multiplicative factor with the Jacobi function of the second kind $Q_n^{a-1, b-2}$ in the variable x related to ρ by

$$-\rho = \frac{2}{1-x} \quad \Leftrightarrow \quad x = \frac{\rho+2}{\rho}.$$

Consequently,

$$[D_\rho(\rho D_\rho)] C(\rho) = 2B_{a,b,n} \frac{(1+\rho)^{b-2}}{\rho^{a+b-1}} P_{n-1}^{a-1, b} \left(\frac{\rho+2}{\rho} \right) Q_n^{a-1, b-2} \left(\frac{\rho+2}{\rho} \right)$$

where

$$B_{a,b,n} = \frac{n!\Gamma(a+b+n-1)}{\Gamma(a+n-1)\Gamma(N+n-1)}.$$

Moreover, recall from [16, Eq. (4.4.2)], that (note that $(\rho+2)/\rho > 1$)

$$Q_n^{a-1, b-2} \left(\frac{\rho+2}{\rho} \right) = \frac{\rho^{a+b-3}}{2^{a+b-4}(\rho+1)^{b-2}} \int_{-1}^1 (1-u)^{a-1} (1+u)^{b-2} \frac{P_n^{a-1, b-2}(u)}{((\rho+2)/\rho) - u} du.$$

As a result

$$\begin{aligned} [D_\rho(\rho D_\rho)] C(\rho) &= B_{a,b,n} \frac{1}{2^{a+b-3}\rho^2} P_{n-1}^{a-1, b} \left(\frac{\rho+2}{\rho} \right) \int_{-1}^1 (1-u)^{a-1} (1+u)^{b-2} \\ &\quad \frac{P_n^{a-1, b-2}(u)}{((\rho+2)/\rho) - u} du \\ &= B_{a,b,n} \frac{1}{2^{a+b-3}\rho^2} \int_{-1}^1 (1-u)^{a-1} (1+u)^{b-2} \left(P_{n-1}^{a-1, b} \left(\frac{\rho+2}{\rho} \right) - P_{n-1}^{a-1, b}(u) \right) \\ &\quad \frac{P_n^{a-1, b-2}(u)}{((\rho+2)/\rho) - u} du \\ &\quad + B_{a,b,n} \frac{1}{2^{a+b-3}\rho^2} \int_{-1}^1 (1-u)^{a-1} (1+u)^{b-2} P_{n-1}^{a-1, b}(u) \frac{P_n^{a-1, b-2}(u)}{((\rho+2)/\rho) - u} du. \end{aligned}$$

Since

$$u \mapsto \frac{1}{((\rho+2)/\rho) - u} \left(P_{n-1}^{a-1,b} \left(\frac{\rho+2}{\rho} \right) - P_{n-1}^{a-1,b}(u) \right)$$

is a polynomial of degree $n-2$, then the orthogonality of the Jacobi polynomials entails

$$\begin{aligned} [D_\rho(\rho D_\rho)] C(\rho) &= B_{a,b,n} \frac{1}{2^{a+b-3} \rho^2} \int_{-1}^1 (1-u)^{a-1} (1+u)^{b-2} P_{n-1}^{a-1,b}(u) \frac{P_n^{a-1,b-2}(u)}{((\rho+2)/\rho) - u} du \\ &= B_{a,b,n} \frac{1}{2^{a+b-3}} \int_{-1}^1 (1-u)^{a-1} (1+u)^{b-2} P_{n-1}^{a-1,b}(u) \frac{P_n^{a-1,b-2}(u)}{\rho(\rho+2-\rho u)} du. \end{aligned}$$

Writing

$$\frac{1}{\rho(\rho+2-\rho u)} = \frac{1}{2} \left[\frac{1}{\rho} - \frac{(1-u)}{\rho+2-\rho u} \right], \quad u \in [-1, 1],$$

and using again the orthogonality of Jacobi polynomials, we get

$$[D_\rho(\rho D_\rho)] C(\rho) = -\frac{B_{a,b,n}}{2^{a+b-2}} \int_{-1}^1 (1-u)^a (1+u)^{b-2} \frac{P_{n-1}^{a-1,b}(u) P_n^{a-1,b-2}(u)}{(\rho(1-u)+2)} du$$

which makes sense for $\rho = 0$. A first integration with respect to ρ gives

$$\begin{aligned} [\rho D_\rho] C(\rho) &= -\frac{B_{a,b,n}}{2^{a+b-2}} \int_{-1}^1 (1-u)^{a-1} (1+u)^{b-2} P_{n-1}^{a-1,b}(u) P_n^{a-1,b-2}(u) \\ &\quad [\ln(\rho(1-u)+2) - \ln 2] du \end{aligned}$$

and a second one leads to

$$C(\rho) = -\frac{B_{a,b,n}}{2^{a+b-2}} \int_{-1}^1 (1-u)^{a-1} (1+u)^{b-2} P_{n-1}^{a-1,b}(u) P_n^{a-1,b-2}(u) \left\{ \int_0^\rho \frac{\ln(v(1-u)/2+1)}{v} dv \right\} du.$$

Performing the variable changes $u \mapsto 1-2u$ in the last expression, we end up with

$$C(\rho) = -B_{a,b,n} \int_0^1 u^{a-1} (1-u)^{b-2} P_{n-1}^{a-1,b}(1-2u) P_n^{a-1,b-2}(1-2u) \left\{ \int_0^\rho \frac{\ln(vu+1)}{v} dv \right\} du$$

for any $\rho \in [0, 1[$. By analytic continuation, this formula extends to the cut plane $\mathbb{C} \setminus (-\infty, 0)$ and is in particular valid for $\rho \geq 0$. Specializing it to $a = m_t - m_r + 1$, and $b = m - m_t - m_r + 1$ completes the proof of the theorem.

APPENDIX C

PROOF OF THEOREM 2

Perform the variable change $\rho \mapsto b\rho$ in the definition of $C(m_t, m_r, m, \rho)$:

$$\begin{aligned} C(b\rho) &= Z_{a,b,n}^{-1} \int \ln \left(\prod_{i=1}^n (1 + b\rho\lambda_i) \right) \prod_{i=1}^n \lambda_i^{a-1} (1 - \lambda_i)^{b-1} V(\lambda)^2 \mathbf{1}_{\{0 < \lambda_1 < \dots < \lambda_n < 1\}} d\lambda \\ &= \frac{Z_{a,b,n}^{-1}}{b^{(an+n(n-1))}} \int \ln \left(\prod_{i=1}^n (1 + \rho\lambda_i) \right) \prod_{i=1}^n \lambda_i^{a-1} \left(1 - \frac{\lambda_i}{b} \right)^{b-1} V(\lambda)^2 \mathbf{1}_{\{0 < \lambda_1 < \dots < \lambda_n < b\}} d\lambda. \end{aligned}$$

On the other hand, our obtained expression for the ergodic capacity together with the variable change $v \mapsto bv$ entail:

$$C(b\rho) = -\frac{B_{a,b,n}}{b^a} \int_0^1 u^{a-1} \left(1 - \frac{u}{b}\right)^{b-2} P_{n-1}^{a-1,b} \left(1 - \frac{2u}{b}\right) P_n^{a-1,b-2} \left(1 - \frac{2u}{b}\right) \left\{ \int_0^\rho \frac{\ln(vu+1)}{v} dv \right\} du$$

Now

$$\lim_{b \rightarrow \infty} \frac{B_{a,b,n}}{b^a} = \frac{n!}{\Gamma(a+n-1)}$$

and similarly

$$\lim_{b \rightarrow \infty} \frac{Z_{a,b,n}^{-1}}{b^{n(a+n-1)}} = n! \prod_{i=1}^n \frac{1}{\Gamma(i+1)\Gamma(a+i-1)} = \prod_{i=1}^n \frac{1}{\Gamma(i)\Gamma(a+i-1)}.$$

Moreover, the limiting transition (6) yields

$$\begin{aligned} \lim_{b \rightarrow \infty} P_{n-1}^{a-1,b} \left(1 - \frac{2u}{b}\right) &= L_{n-1}^{a-1}(u) \\ \lim_{b \rightarrow \infty} P_n^{a-1,b-2} \left(1 - \frac{2u}{b}\right) &= L_n^{a-1}(u). \end{aligned}$$

As a result,

$$\begin{aligned} \lim_{b \rightarrow \infty} C(b\rho) &= \prod_{i=1}^n \frac{1}{\Gamma(i)\Gamma(a+i-1)} \int \ln \left(\prod_{i=1}^n (1 + \rho\lambda_i) \right) \prod_{i=1}^n \lambda_i^{a-1} e^{-\lambda_i} V(\lambda)^2 \mathbf{1}_{\{0 < \lambda_1 < \dots < \lambda_n\}} d\lambda \\ &= -\frac{n!}{\Gamma(a+n-1)} \int_0^{+\infty} u^{a-1} e^{-u} L_{n-1}^{a-1}(u) L_n^{a-1}(u) \left\{ \int_0^\rho \frac{\ln(vu+1)}{v} dv \right\} du. \end{aligned}$$

Finally,

$$\prod_{i=1}^n \frac{1}{\Gamma(i)\Gamma(a+i-1)}$$

is the normalization constant of the density of the joint distribution of the ordered eigenvalues of a complex Wishart matrix [14]. The theorem is proved.

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